4.5: Laws of Exponents ad Logarithm

Laws of Exponents: Let a and b be positive numbers and let x and y be real numbers. Then,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $b^0 = 1$
5. $b^1 = b$
6. $(ab)^x = a^x b^x$
7. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Laws of Logarithms

If m and n are positive numbers and $b > 0$, $b \neq 1$, then

1. $\underbrace{\log_b(mn)}_{\text{log(a product)}} = \underbrace{\log_b(m) + \log_b(n)}_{\text{sum of log}s}$ — Log of a product is the sum of logs
2. $\underbrace{\log_b\left(\frac{m}{n}\right)}_{\text{log(a quotient)}} = \underbrace{\log_b(m) - \log_b(n)}_{\text{difference of log}s}$ — log of a quotient is difference of logs
3. $\log_b(m^n) = n\log_b(m)$ — log of an exponential is the product of exponent × log of base of the exponential
4. $\log_b(1) = 0$ — log of one is zero
5. $\log_b(b) = 1$ — log of the base is one
6. $\log_c(x) = \dfrac{\log_b(x)}{\log_b(c)}$ — Change of base formula

Fun Facts:

Numbers 1-6 of laws of logarithm are consequences of laws of exponents.

- **Show "Law of exponent 1 $\Longrightarrow$ Law of Logarithm 1":**

  (1) $\underbrace{mn = b^{\log_b(mn)}}_{\text{By Inverse Property}}$

  (2) $\underbrace{mn = \overbrace{b^{\log_b(m)}}^{m}\,\overbrace{b^{\log_b(n)}}^{n}}_{\text{By Inverse Property on Each Factor}}$

  (3) $\underbrace{mn = \overbrace{b^{\log_b(m)+\log_b(n)}}^{b^{\log_b(m)}b^{\log_b(n)}}}_{\text{By Exponential Property 1}}$

  (4) By (1) and (3): $b^{\log_b(mn)} = b^{\log_b(m)+\log_b(n)}$. That is, $\log_b(mn) = \log_b(m) + \log_b(n)$.

- To **combine** two or more logs, make sure that the coefficients become exponents inside the log, then use the log of sum and difference law. We combine to **solve equations**.

- To **expand** a log, use the law of sum and differences first, and then use the law of exponents and simplify using inverse function property. We expand to **evaluate** and in **future calculus techniques.**

- You will see applications of combining or expanding in the future sections.

1. For $a > 0, b > 0, c > 0$, $\left(81a^{-4}b^8c^4\right)^{\frac{1}{4}} =$

(A) $3b^2$ (B) $\dfrac{b^2c}{3a}$ (C) $\dfrac{3b^2c}{a}$ (D) $\dfrac{9b^2c}{a}$ (E) $\dfrac{b^2c}{81a}$

2. $\sqrt[5]{3}\sqrt[4]{3} =$

(A) 3 (B) $\sqrt[20]{3}$ (C) $\sqrt[20]{3^4}$ (D) $\sqrt[20]{3^5}$ (E) $\sqrt[20]{3^9}$

3. $\ln(\sqrt{x^2+16}) - \ln(\sqrt{x})$ is equivalent to

(A) $\frac{1}{2}\ln\left(\frac{x^2+16}{x}\right)$ (B) $\dfrac{\ln(\sqrt{x^2+16})}{\ln(\sqrt{x})}$ (C) $\ln\left(\dfrac{x+4}{\sqrt{x}}\right)$ (D) $\ln(4)$ (E) $\ln\left(\sqrt{x^2+16} - \sqrt{x}\right)$

4. When **expanded** using the laws of logarithms, $\ln\left(\dfrac{a}{b^2\sqrt{c}}\right) =$

(A) $\ln a - 2\ln b - \frac{1}{2}\ln c$ (B) $a - 2b - \frac{1}{2}c$ (C) $\ln(a - 2b + \frac{1}{2}c)$ (D) $\ln a - 2\ln b + \frac{1}{2}\ln c$

5. Simplify: (A) $e^{7\ln x}$ (B) $\ln(e^{t^2})$

6. Simplify the following expressions.

(A) $\dfrac{2^{y+1}e^{3x}}{e^x 2^y}$

(B) $(2^{2x}3^{3x}2^{2x-1})^3$

7. **Expand** the following expressions.

(A) $\ln\left((e^{3x}e^{x+1})^{1/3}\right)$

(B) $\ln\left(\dfrac{\sqrt{e^x}\sqrt{2^x}}{2e}\right)$

8. Use the laws of logarithms to **combine** each expression into single logarithm.

(A) $\frac{1}{2}\ln(x) + 8\ln(y) - 5\ln(z)$ (B) $2\ln(x+3) - \frac{1}{3}\ln(x^2+x) + \ln 8$ (C) $8\ln(x+6) - \frac{1}{2}\ln(x+8) + 3\ln(x)$.

9. Use the laws of logarithms to **expand** and simplify the expression. $\log(x(x^5+5)^{-1/2}) =$

10. Factor and simplify the following function as much as possible. Then find the **zeros** and the **domain** of the function.

$$f(x) = 30(x+3)^2(x-2)^{-\frac{2}{3}} - 6(x+3)(x-2)^{\frac{1}{3}}$$

A few Videos:

1. https://youtu.be/n6Nvm8j6P6A
2. https://youtu.be/FPVvZwZnuPw